Section 8-1, Mathematics 104

Systems of Linear Equations

A little micro economics:

If you were looking to buy a TV, you might be concerned about the price.

If the TV cost \$20 you would probably jump at the chance.

If the TV cost \$20,000 you would probably forget about it.

When dealing with a large sample of people, there is a function describing how many people would buy a TV at some price.

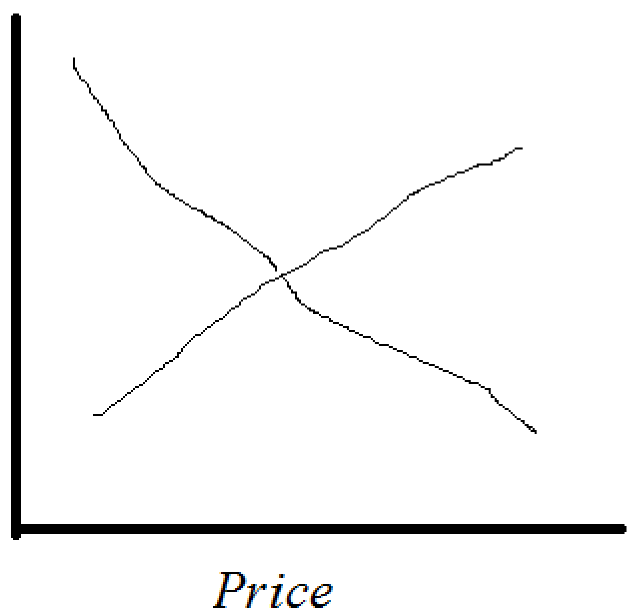
If we graphed this function it might look something like this

Note that the function might be approximated, at least in some parts of the graph by a line. This is called a **demand** curve.

On the other end of the sale, if you were a manufacturer, the number of units you might be willing to build might depend on the price you could get for each unit. The more you get per unit, the more you might be willing to build. The graph of this function might look like this.

This is called a supply curve.

Note that the axes on both graphs are the same. If we place one on top of the other we get this.



The place where the two lines meet represents the place where supply meets demand.

So being able to calculate this point might be useful to a manufacturer. Build less and you lose sales and profit. Build to many and you have unsold stock on your shelf.

This brings us to the subject of solving two linear equations in two unknowns.

If we have two linear functions

$$ax + by = c$$
$$dx + ey = f$$
$$or$$
$$y = ax + b$$
$$y = cx + d$$

We want to know how to find the point (x,y) that will satisfy both equations and therefore represent the intersection point on the graph of the two functions.

Finding the intersection of two linear equations in two unknowns

It turns out that finding the intersection of two equations in two unknowns is so important that there are multiple ways to accomplish find the point of intersection, or the solution.

We will look at 5 different methods

1) Graphing

2) Substitution

3) Elimination

4) Matrices and Gaussian elimination

5) Cramer's rule

It sounds a bit much to look at 5 different ways of doing the same thing, but there is a reason.

Graphing is usually not a very accurate way to find a solution, however understanding how we solve with graphs may give you some visual intuition about the problem.

Substitution and Elimination are the standard methods taught in high school. They are actually not the easiest, however the techniques are extendible to other types of problems that are more complex.

Learning about matrices it itself a large area in math. We will focus on how it can make solving two equations in two unknowns easy. This method extends nicely to 3 equations in 3 unknowns and beyond.

Cramer's rule is interesting because it provides a formula that solves the problem directly. This can be very useful if you are programming a computer solve a problem, but it also extends to more than 2 variables and 2 equations.

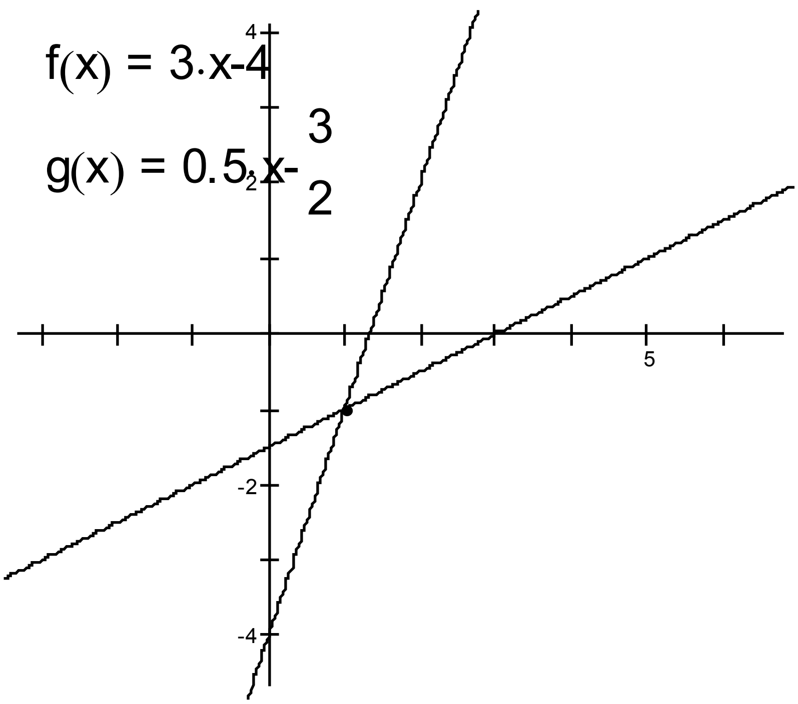
Finding the solution by Graphing

We can always find the solution by graphing. This method takes some skill and can provide an imprecise answer.

Here's an example:

$$y = 3x - 4$$
$$2y = x - 3$$

Dividing by w the second equation becomes $y = \frac{x}{2} - \frac{3}{2}$ in slope intercept form.



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Looking at the graph it appears that the intersection point is about (1,-1).

Plugging these in we find -1=3-4 and -2=1-3, both correct.

This is an exact answer but it won't always turn out this way.

Solving by substitution

The key to solving by substitution is to have at least one of the equations in the form

$y=$ or $x=$

We can then substitute one equation into the other, removing a variable. Since the new equation will be in just one unknown we can solve for that variable.
Once we know either the x or y value of the solution, it is easy to plug in and find the other value.

Example:

$$-x+y=1$$
$$2x+y=-2$$

We can solve either of these equations for y so we choose the first

$$y=x+1$$

We substitute $x+1$ for y in the second equation.

Now we have a simple equation in one unknown which we are well equipped to solve.

$$2x+(x+1)=-2$$
$$3x+1=-2$$
$$3x=-3$$
$$x=-1$$

Having found x we substitute it into the first equation

$$-(-1)+y=1$$
$$1+y=1$$
$$y=0$$

So the solution is (-1,0)

We can plug this into the 2nd equation as a check.

$$2(-1)+0=-2$$
$$-2=-2$$

Solution Possibilities

At this point we should consider what the possible solutions might look like.
From geometry we know that two distinct lines will meet at, at most one point.
They might not meet at all if they are parallel.
The third possibility is that the two lines are the same in which case there will be an infinite number of solutions.

Examples: (Parallel)

$$y = 2x + 3$$
$$y = 2x + 4$$

These two lines are parallel.
If we try to solve by substitution we get

$$2x + 3 = 2x + 4$$
$$3 = 4$$

Since 3 is obviously not equal to 4, there are no solutions.

Example: (Same line)
If we have two equations for the same line:

$$2x + 3y = 4$$
$$4x + 6y = 8$$

If we try to solve by substitution we get

$$2x + 3y = 4$$
$$y = \frac{4 - 2x}{3}$$
$$4x + 6\left(\frac{4 - 2x}{3}\right) = 8$$
$$4x + 8 - 4x = 8$$
$$8 = 8$$

Because 8=8 is true, there will be an infinite number of solutions because the lines are identical.

Elimination

Elimination is most useful when just adding or subtracting both sides of the two equations eliminates a variable.

Example:

$$2x+3y=5$$
$$-2x+y=3$$

Note that we can add these two equations easily and immediately eliminate x

$$4y=8 \rightarrow y=2$$

Sometimes we can get almost the same simplicity by multiplying one equation by a constant.

Example:

$$3x-2y=4$$
$$2x+y=5$$

Note that if I multiply the 2nd equation by 2 and add

$$3x-2y=4$$
$$4x+2y=10$$

We get

$$7x=14 \rightarrow x=2$$

In the worst case you need to multiply both equations

Example:

$$3x + 5y = 11$$
$$2x - 7y = 5$$

Here our best bet it to multiply the first equation by 3, the second by -3 and add

$$6x + 10y = 22$$
$$-6x + 21y = -15$$

leaving

$$31y = 7 \to y = \frac{7}{31}$$

Note that the solutions are not always nice, but they will always be a rational number.

Some problems to try in class:

$$x + 2y = 8$$
$$2x - 3y = -19$$

$$2x - y = 7$$
$$4x - 3y = 9$$

$$3x + y = 7$$
$$4x + y = 13$$

$$4x + 3y = 11$$
$$4x - y = 23$$

$$5x - 2y = -25$$
$$3x + y = -4$$

$$\frac{x}{2} + \frac{y}{3} = 6$$
$$3x - 2y = 12$$